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A Theoretical Study of Navigation Precision as a Function of Observational Errors

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INTRODUCTION

As a part of the experimental program of the Edwards Street Laboratory, it was necessary to locate equipment in the waters of a harbor with precision, and both to follow the course of boats in relation to this equipment and to maneuver boats over pre-chosen courses. Precise methods of navigation were accordingly of interest. This theoretical report was prepared as a guide to the degree of precision of location and of navigation which might be attained with three different methods of measurement.

ABSTRACT

An important question for any navigational system is the following. For a pre-assigned uncertainty in position what is the maximum error which can be tolerated in measuring the coordinates of position? Three types of navigational systems are considered in this report: a) Position determined by measuring at two fixed stations, a known distance apart, the angles subtended by the vessel and the baseline connecting the stations, b) Position determined by measuring at a single station the range and angle subtended by the vessel relative to a fixed direction, c) Position determined by measuring the ranges from two fixed stations a known distance apart. For each case curves have been computed which relate the uncertainty in position to the maximum permissible error of the position parameters. This information also has been presented in the form of maximum error contours which can be scaled and superimposed on a chart for direct reading.

This report was edited and the abstract prepared by A. Voorhis.

The subject of this report arises naturally out of the problem of determining the position of a ship or other object near the shore by the use of fixed reference stations on shore. Of the various combinations of position parameters(measured quantities,such as angle, distance,etc.),we have restricted our attention to the following:

1. Consider two fixed shore stations. Parameters: the two angles included between the line of sight from each station to the ship and a suitable fixed direction (usually the base line). Figure 1 .

2. One shore station. Parameters: the distance from the station to the ship and the angle included between the line of sight and a suitable fixed direction. Figure 7 .

3. Two shore stations. Parameters: the two distances from each station to the ship. Figure 11 .

In each of these three case,we shall define the error of position and show how this error varies as a function of position and of assumed errors in the position parameters. However,the principal purpose of this report is to treat the inverse problem: given a

preassigned maximum allowable error of position, with what accuracy need one measure the position parameters in order to obtain a fix which remains within the given allowable error of position? We purposely neglect the possibility that one or several of the position parameters may be measured with extreme accuracy; our main interest lies indetermining -- as a function of position -- the maximum error that one is allowed to make in measuring all of the position parameters. An implicit assumption here is that the measuring instruments have an error distribution curve which is finite in extent, that is, the probablity of an instrument error greater than some fixed constant is zero,

We realize that our approach to the problem may be "wasteful" in a sense; but it is hoped that the results will justify our point of view.

Case 1. Consider the following figure.

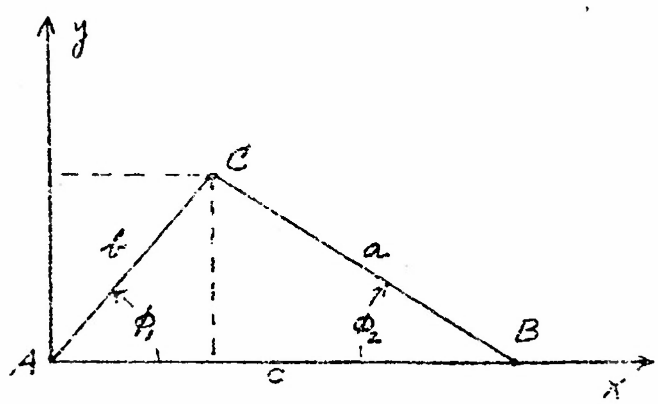


Figure 1.

ϕ_1, ϕ_2 :measured angles

A,B :fixed reference stations

C :object whose position is to be determined

c :distance between stations

By using the law of sines and by noticing that angle

$$ACB = \pi - (\phi_1 + \phi_2)$$

and that $\sin(\pi - \phi_1 - \phi_2) = \sin(\phi_1 + \phi_2)$,we obtain

$$b = c \frac{\sin \phi_2}{\sin(\phi_1 + \phi_2)} \tag{1}$$

Then the x and y coordinates of C are easily seen to be

$$x = c \frac{\sin \phi_2 \cos \phi_1}{\sin(\phi_1 + \phi_2)}$$

$$y = c \frac{\sin \phi_1 \sin \phi_2}{\sin(\phi_1 + \phi_2)} \tag{2}$$

Thus $x = f(c, \phi_1, \phi_2)$ and $y = g(c, \phi_1, \phi_2)$. The errors in the x and y coordinates of C in terms of the errors in c , ϕ_1 and ϕ_2 are

$$dx = \frac{\partial f}{\partial c} dc + \frac{\partial f}{\partial \phi_1} d\phi_1 + \frac{\partial f}{\partial \phi_2} d\phi_2$$

$$dy = \frac{\partial g}{\partial c} dc + \frac{\partial g}{\partial \phi_1} d\phi_1 + \frac{\partial g}{\partial \phi_2} d\phi_2 \tag{3}$$

Here, we may assume that the distance c can be measured as accurately as we please; therefore we may set $dc = 0$ in equation (3)

Now

$$dx = -c \sin\phi_2 \left[\frac{\sin\phi_1 \sin(\phi_1+\phi_2) + \cos\phi_1 \cos(\phi_1+\phi_2)}{\sin^2(\phi_1+\phi_2)}\right] d\phi_1$$

$$+ c \cos\phi_1 \left[\frac{\cos\phi_2 \sin(\phi_1+\phi_2) - \sin\phi_2 \cos(\phi_1+\phi_2)}{\sin^2(\phi_1+\phi_2)}\right] d\phi_2$$

$$= \frac{c}{\sin^2(\phi_1+\phi_2)} \left[-\sin\phi_2 \cos\phi_2 \, d\phi_1 + \sin\phi_1 \cos\phi_1 \, d\phi_2\right]$$

and

$$dy = c \sin\phi_2 \left[\frac{\cos\phi_1 \sin(\phi_1+\phi_2) - \sin\phi_1 \cos(\phi_1+\phi_2)}{\sin^2(\phi_1+\phi_2)}\right] d\phi_1$$

$$+ c \sin\phi_1 \left[\frac{\cos\phi_2 \sin(\phi_1+\phi_2) - \sin\phi_2 \cos(\phi_1+\phi_2)}{\sin^2(\phi_1+\phi_2)}\right] d\phi_2$$

$$= \frac{c}{\sin^2(\phi_1+\phi_2)} \left[\sin^2\phi_2 \, d\phi_1 + \sin^2\phi_1 \, d\phi_2\right]$$

We shall term the error of position of C as $dS = \sqrt{dx^2+dy^2}$.
Geometrically, the error of position, dS, is exactly what we might expect: it is the length of the line CC' in figure 2.

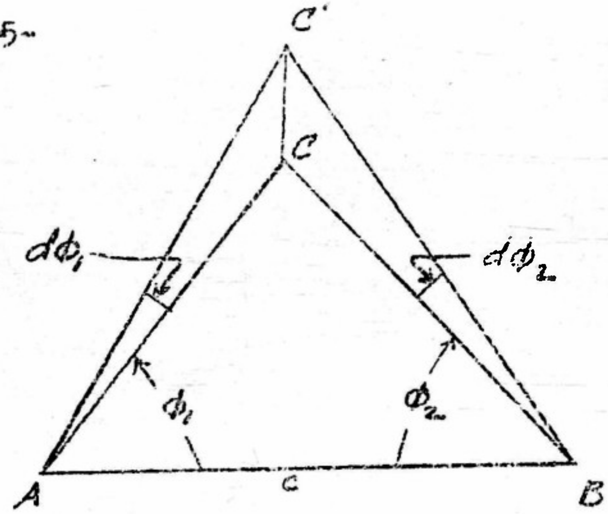


Figure 2.

By straightforward calculation together with simple trigonometric identities, we have

$$ds^2 = \frac{c^2}{\sin^4(\phi_1+\phi_2)} \times \tag{4}$$

$$\times \left[\sin^2\phi_2 \, d\phi_1^2 + \sin^2\phi_1 \, d\phi_2^2 - 2 \sin\phi_1 \sin\phi_2 \cos(\phi_1+\phi_2) \, d\phi_1 \, d\phi_2\right]$$

Equation (4), then, gives the error of position as a function of c, ϕ_1, ϕ_2, $d\phi_1$ and $d\phi_2$; however, the expression on the right is rather complicated and difficult to interpret as it stands. We shall make the following simplification which, in our opinion, is not unrealistic. Letting $d\phi_1 = \pm d\phi_2$ equation (4) becomes

$$\frac{c^2 d\phi_1^2}{ds^2} = \frac{\sin^4(\phi_1+\phi_2)}{\sin^2\phi_2 + \sin^2\phi_1 \pm 2 \sin\phi_1 \sin\phi_2 \cos(\phi_1+\phi_2)} \tag{5}$$

We should like the term $\frac{c\,d\phi_1}{ds}$ to give the maximum error that is allowed in measuring ϕ_1 and ϕ_2, for a given separation of the reference stations and for a given maximum allowable error of position. Equation (5) is practically useless for this p urpose, since the denominator of the right-hand member is not single-valued. In order to rectify this situation, we replace the $(\pm)$ by $(+)$ and take the absolute value of $\cos(\phi_1+\phi_2)$ thus making the denominator single-valued and giving $\frac{c^2\,d\phi_1^2}{ds^2}$ its smaller value. This change insures that $d\phi_1$ is always within the allowable error range in the measurement of ϕ_1 (or ϕ_2). The final expression which we seek is then

$$\frac{c^2\,d\phi_1^2}{ds^2} = \frac{\sin^4(\phi_1+\phi_2)}{\sin^2\phi_2+\sin^2\phi_1+\sin\phi_1\sin\phi_2\,|\cos(\phi_1+\phi_2)|} \qquad (6)$$

Equation (6) is still somewhat complicated and so several graphs have been prepared to show more clearly the variation of $\frac{c\,d\phi_1}{ds}$ with the position of the object. Also, a contour chart is included showing those regions of the harbor (relative to the reference stations) where $\frac{c\,d\phi_1}{ds}$ assumes certain constant values. Each graph is fully explained on the page preceding it.

Figures 3 and 4

Here we see the variation of $\frac{c\,d\phi_1}{ds}$ as a function of one of the measured angles with the other angle as a parameter. Note that either angle may be used as parameter since equation (6) is symmetric in ϕ_1 and ϕ_2. Figure 4 on the following page is a magnification of Figure 3 for the range $0 \leq \frac{c\,d\phi_1}{ds} \leq 0.4$.

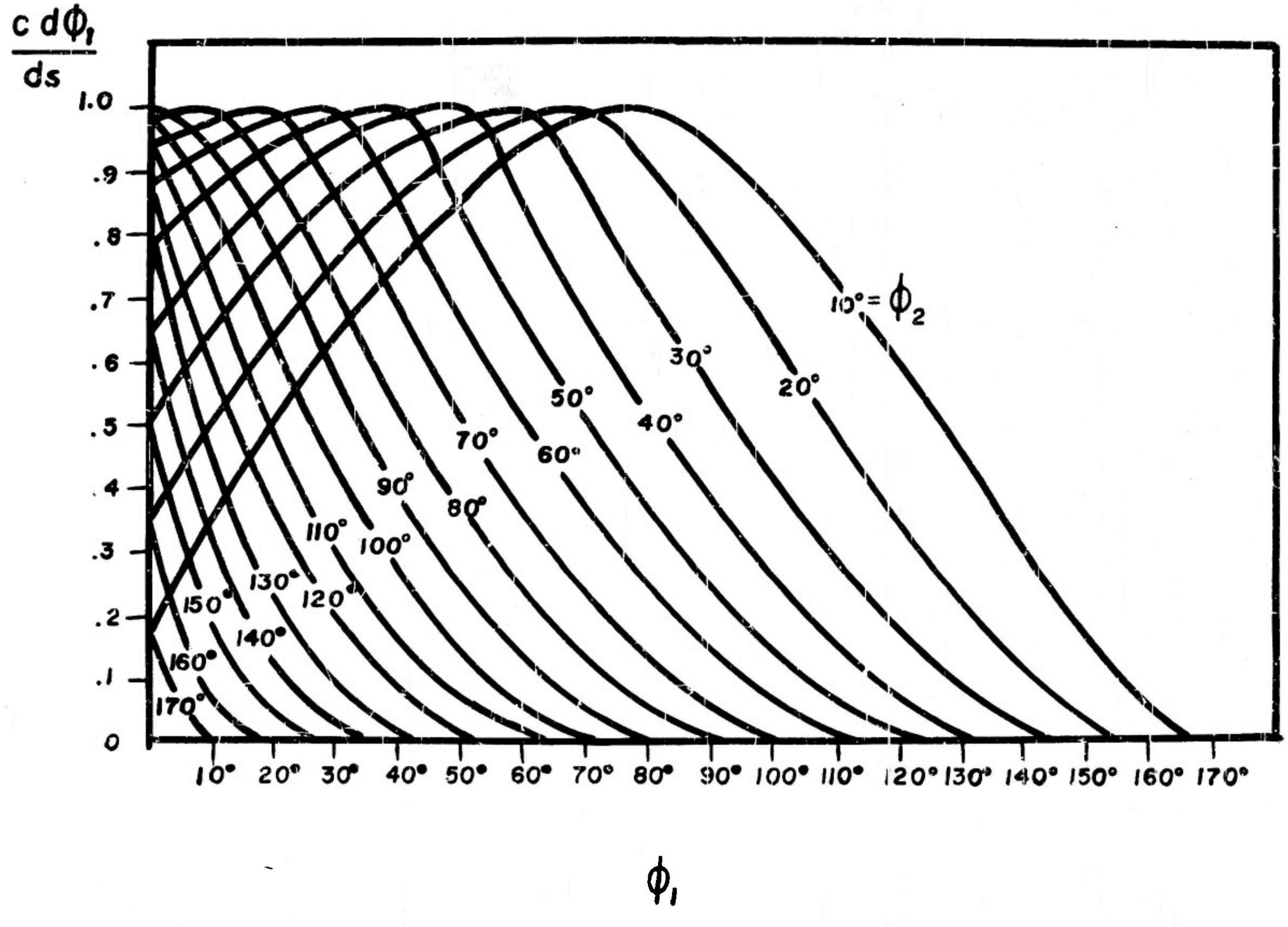
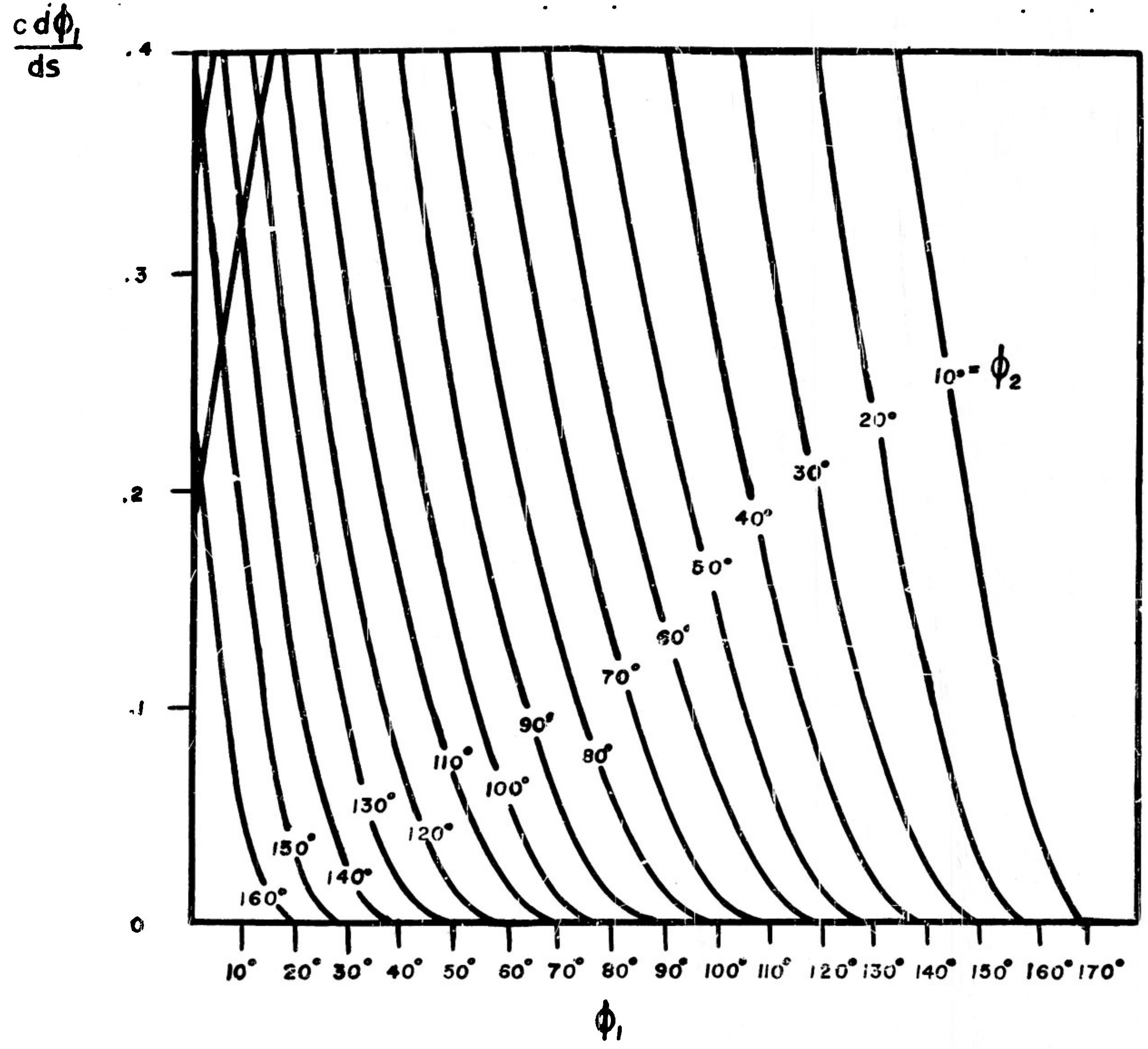

c dϕ1 / ds
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10°= ϕ2
20°
30°
40°
50°
60°
70°
80°
90°
100°
110°
120°
130°
140°
150°
160°
170°
10° 20° 30° 40° 50° 60° 70° 80° 90° 100° 110° 120° 130° 140° 150° 160° 170°
ϕ1

Figure 3

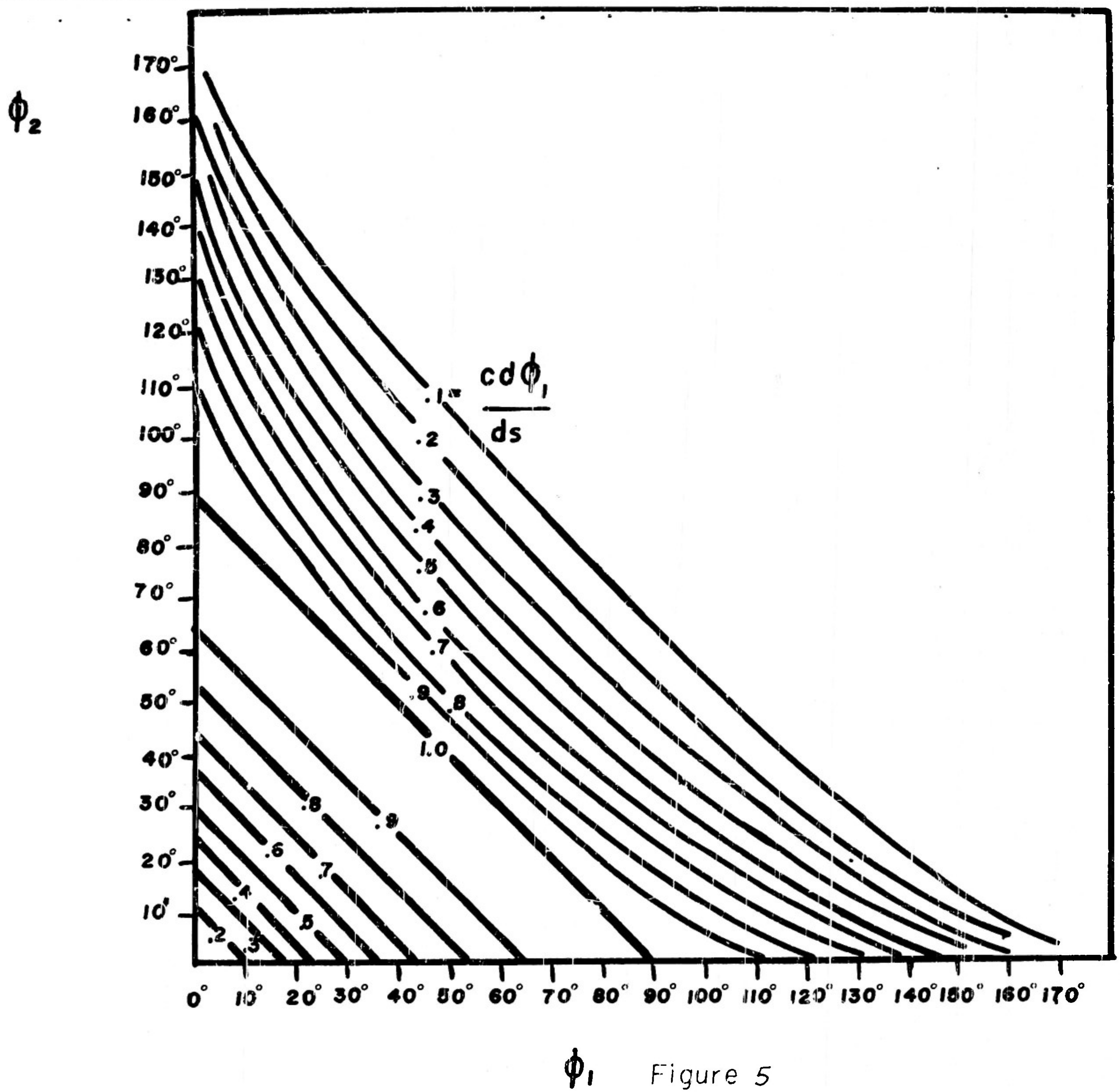


$\frac{c\,d\phi_1}{ds}$
.4
.3
.2
.1
0
$10° = \phi_2$
20°
30°
40°
50°
60°
70°
80°
90°
100°
110°
120°
130°
140°
150°
160°
10° 20° 30° 40° 50° 60° 70° 80° 90° 100° 110° 120° 130° 140° 150° 160° 170°
ϕ_1

Figure 4

Figures 5 and 6

Figure 5 shows the relation between the measured angles ϕ_1 and ϕ_2 for various of $\frac{c\,d\phi_1}{dS}$. Figure 6 shows the partitioning of the area under consideration into equal-error regions, that is, those regions in which the maximum allowable error of measurement, $d\phi_1$ $(= d\phi_2)$, is the same for given c and dS. Note that the curve labled 1.0 is a portion of a circle passing through the reference stations. On this curve, the lines of sight intersect at right angles and maximum allowable error in ϕ_1 (and ϕ_2) is the largest.



ϕ_2
$\frac{c\,d\phi_1}{ds}$
.1 =
.2
.3
.4
.5
.6
.7
.8
.9
1.0
.2
.3
.4
.5
.6
.7
.8
.9
170°
160°
150°
140°
130°
120°
110°
100°
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
100°
110°
120°
130°
140°
150°
160°
170°
ϕ_1

Figure 5

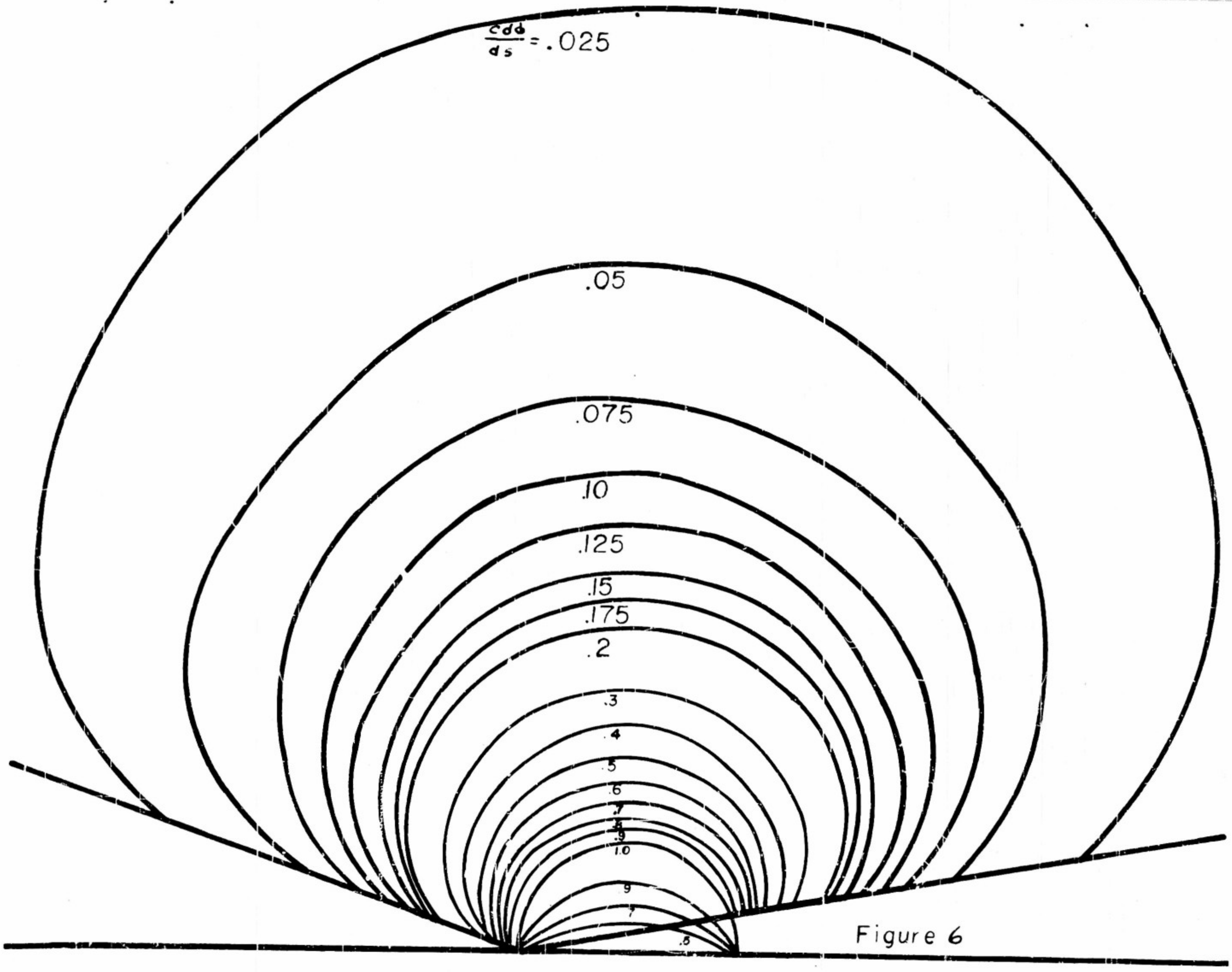
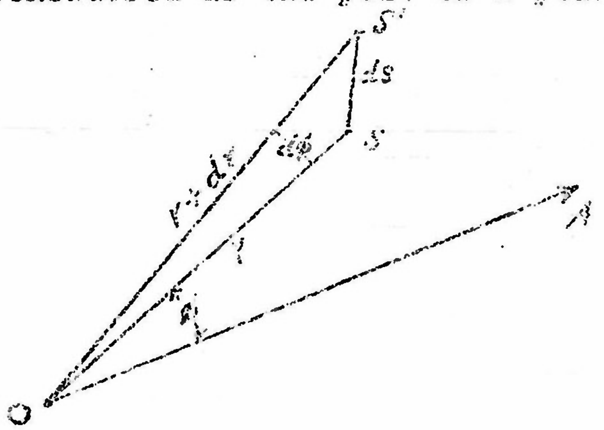


Figure 6

<u>Case 2.</u> If the position of an object is determined relative to one fixed shore station with angle and distance as position parameters, the analysis of position error is particularly easy since the reference station may be considered as the pole of a polar coordinate system.



ϕ :measured angle

r:measured distance

O;reference station

S:ship

OA ;fixed direction

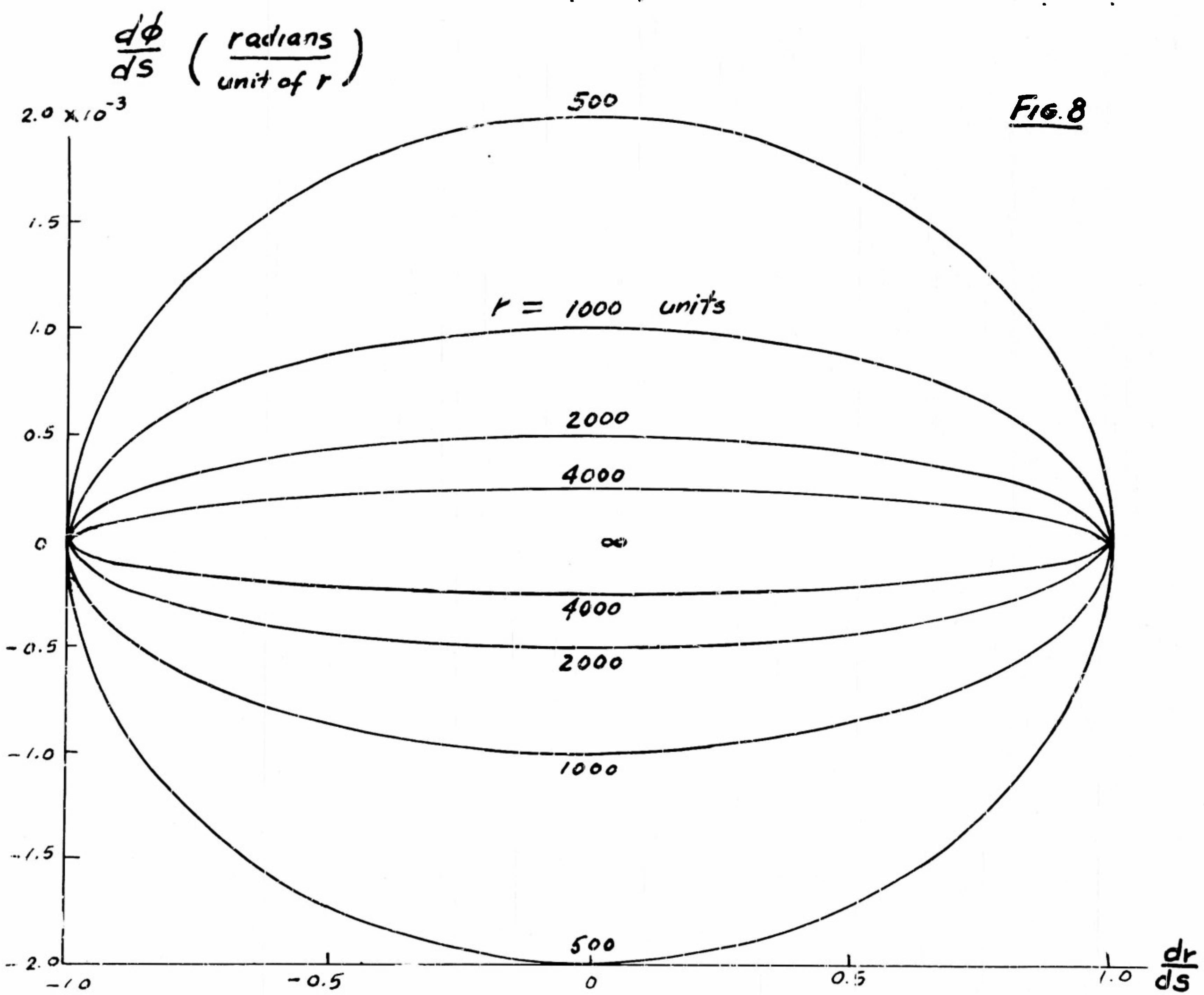
Figure 7.

The error of position of S is given by

$$ds = \sqrt{dr^2 + r^2 d\phi^2} \tag{7}$$

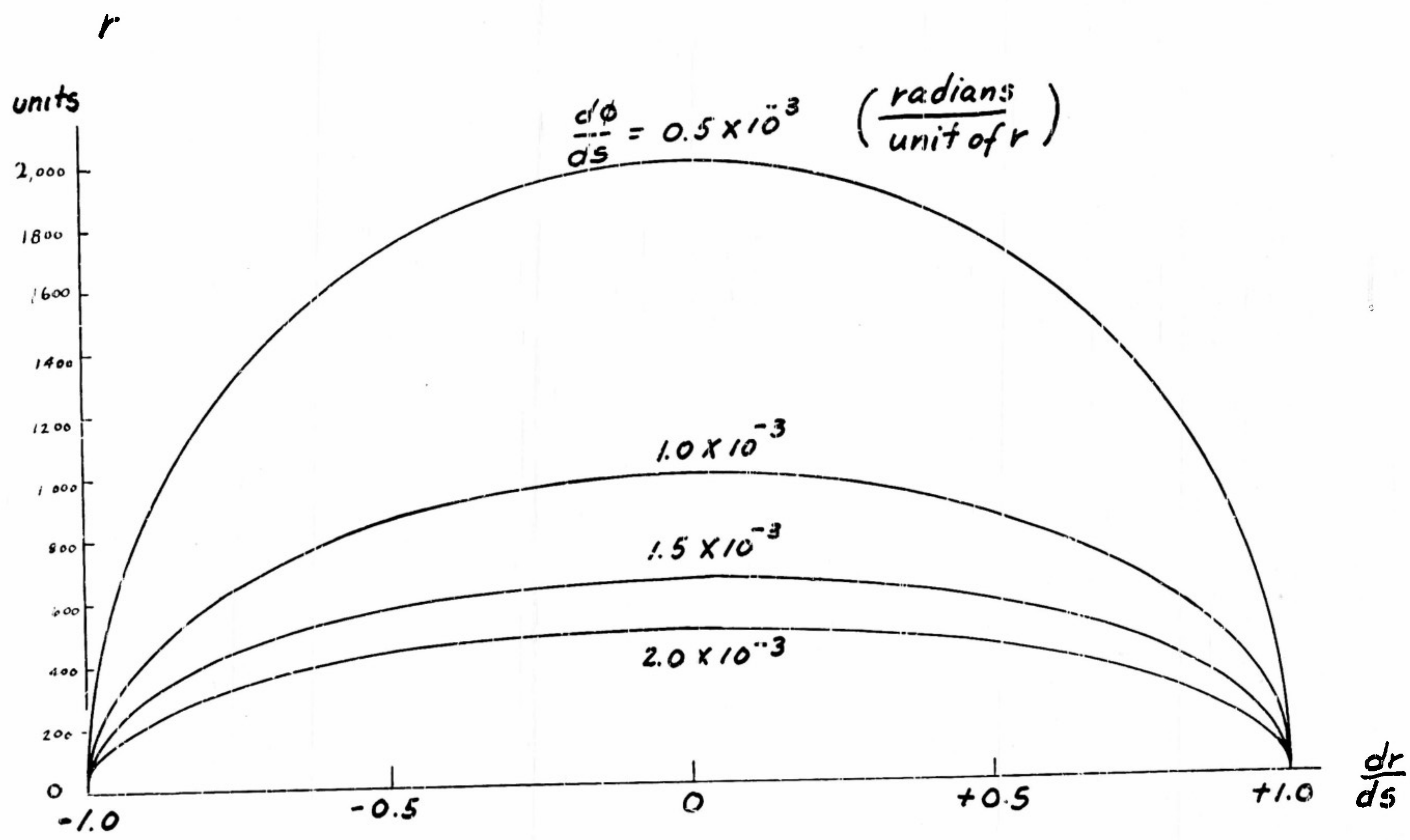
where dr and $d\phi$ are assumed errors in the position parameters r and ϕ . Geometrically, ds is the length of the line SS' in figure 7. We cannot, as in case 1, simplify the above equation by letting $dr = r\,d\phi$, say, since r and ϕ measure quantities of different nature. Instead, we consider one of the variables dr, r and $d\phi$ as a parameter and graph this relation, letting the remaining two quantities act as independent and dependent variable. These graphs comprise figure 8-10.

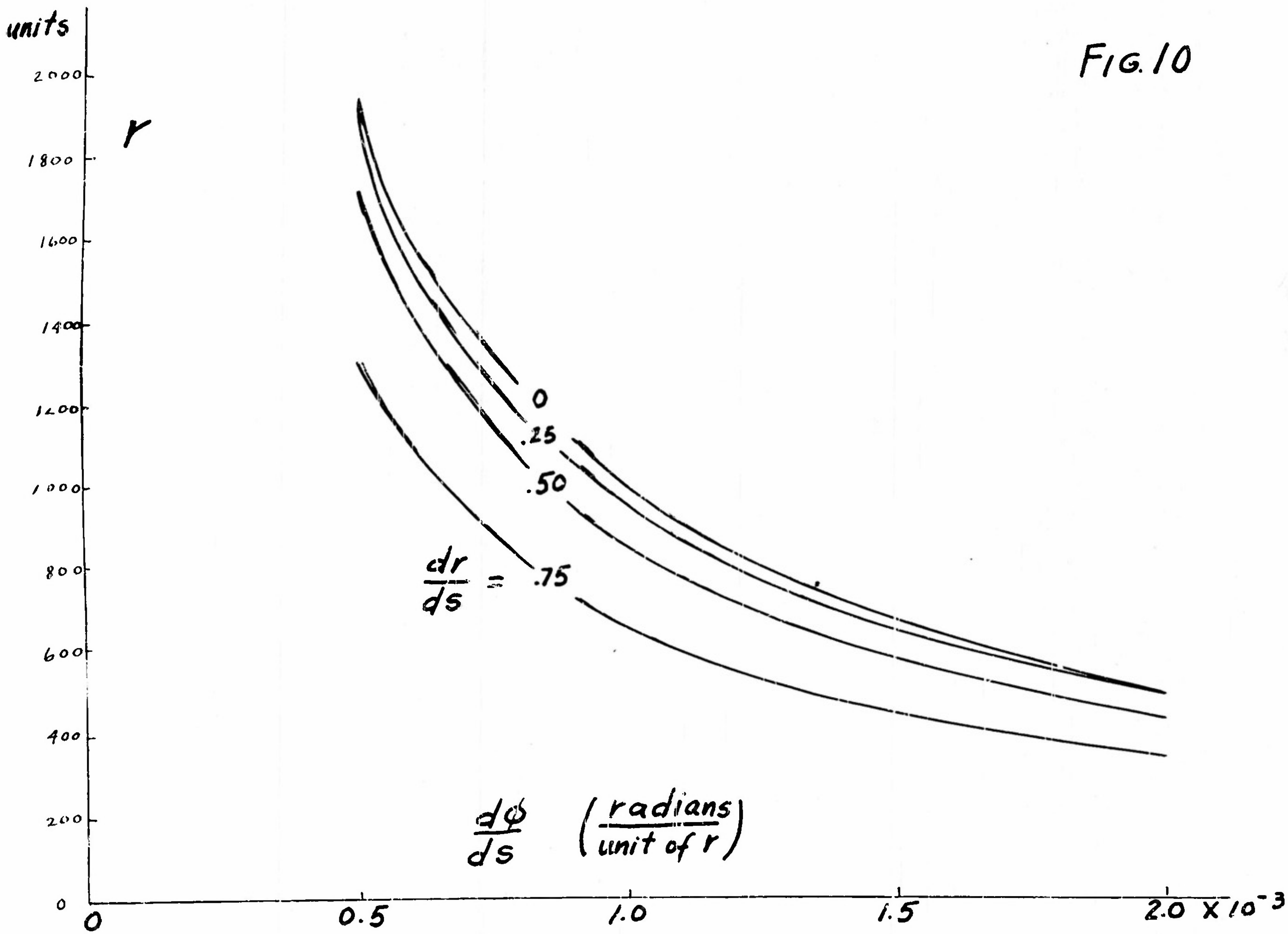
FIG. 8



$\frac{d\phi}{ds}$ $\left(\frac{\text{radians}}{\text{unit of } r}\right)$
2.0×10^{-3}
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0
-1.0
-0.5
0
0.5
1.0
$\frac{dr}{ds}$
500
r = 1000 units
2000
4000
∞
4000
2000
1000
500

FIG. 9


r
units
2,000
1800
1600
1400
1200
1000
800
600
400
200
0
$\frac{d\phi}{ds} = 0.5 \times 10^{-3}$ $\left(\frac{\text{radians}}{\text{unit of } r}\right)$
1.0×10^{-3}
1.5×10^{-3}
2.0×10^{-3}
-1.0
-0.5
0
+0.5
+1.0
$\frac{dr}{ds}$



units
2000
1800
1600
1400
1200
1000
800
600
400
200
0
r
0
.25
.50
.75
$\frac{dr}{ds}$ =
$\frac{d\phi}{ds}$ $\left(\frac{\text{radians}}{\text{unit of } r}\right)$
0
0.5
1.0
1.5
2.0 X 10^{-3}

FIG. 10

We do not give an equal-error diagram for this case; we shall simply describe it. First, let us put equation (7) in the form

$$r = \frac{1}{\left(\frac{d\phi}{ds}\right)} \cdot \sqrt{1 - \left(\frac{dr}{ds}\right)^2} \tag{8}$$

We shall assume that dr does not vary appreciably with distance r but, to fix the ideas, let $dr = K$ (= the largest dr within the range of the measuring instrument). Further, note that ϕ does not occur in (8). Thus, by letting $d\phi$ assume certain fixed values, we obtain concentric circles (with center at the reference station) as the equal-error contours. A different value of K will simply dilate or contract the entire diagram. It should be remarked here that since ds is a preassigned constant, we often write $\frac{d\phi}{ds}$ and $\frac{dr}{ds}$ instead of (dr, ds), respectively. This simply amounts to a change in scale.

<u>Case</u> <u>3</u>. Here we consider two fixed reference stations on shore with two measured distances as position parameters.

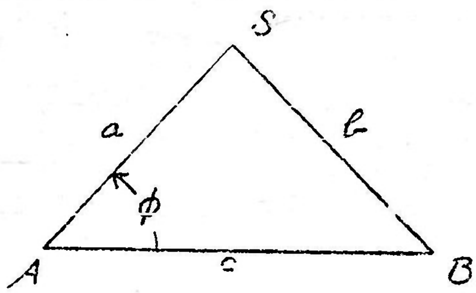


Figure 11

A, B: reference stations

a, b: measured distances

c: separation of stations

Now introduce a system of polar coordinates with center at A. In this system, the point S may be described by the couple (r, ϕ) where $r = a$; consquently, we may define the error of position as $ds^2 = dr^2 + r^2 d\phi^2$. As in the previous two cases we shall find an expression for ds in terms of <u>a</u>, <u>b</u>, <u>da</u>, <u>db</u> and <u>c</u>, where <u>da</u> and <u>db</u> are the errors in <u>a</u> and <u>b</u> respectively. To this end, consider the following analysis.

$$r = a \quad , \quad dr = da$$

$$b^2 = a^2 + c^2 - 2ac \cos\phi \qquad \text{(Low of Cosines)}$$

or

$$\cos\phi = \frac{a^2 + c^2 - b^2}{2ac}$$

[redacted]

and

$$d\phi = \frac{-1}{\sqrt{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}}\left[\left(\frac{1}{2c}-\frac{c}{2a^2}+\frac{b^2}{2a^2c}\right)\underline{da}-\frac{b}{ac}\,\underline{db}\right]$$

Then $\quad ds^2 = dr^2 + r^2 d\phi^2$

$$= \underline{da}^2 + \frac{a^2}{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}\left\{\begin{array}{l}\left(\frac{1}{2c}-\frac{c}{2a^2}+\frac{b^2}{2a^2c}\right)^2\underline{da}^2 \\ -2\left(\frac{1}{2c}-\frac{c}{2a^2}+\frac{b^2}{2a^2c}\right)\underline{da}\,\underline{db}+\frac{b^2}{a^2c^2}\,\underline{db}^2\end{array}\right\}$$

Letting $\underline{da} = \pm\underline{db}$ and rearranging terms, we obtain

$$\left(\frac{da}{ds}\right)^2 = \frac{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}{a^2\left[\left(\frac{1}{2c}-\frac{c}{2a^2}+\frac{b^2}{2a^2c}\right)^2 \pm 2\left(\frac{1}{2c}-\frac{c}{2a^2}+\frac{b^2}{2a^2c}\right)\frac{b}{ac}+\frac{b^2}{a^2c^2}\right]+1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2} \qquad (9)$$

As in case 1, we should like the term $\frac{da}{ds}$ to give the <u>maximum</u> error that is allowed in measuring $\underline{a}$ and $\underline{b}$ for a given maximum allowable error of position.

Thus, in order to make $\frac{da}{ds}$ as small as possible, we choose in equation (9) the (+) when $a^2+b^2 \geq c^2$ and the (—) when $a^2+b^2 < c^2$.

[redacted]

If we choose the $(+)$ and let $\lambda \equiv \frac{a}{c}$ and $\mu \equiv \frac{b}{c}$

then straightforward calculations show that

$$\left(\frac{da}{ds}\right)^2 = \frac{4\lambda^2 - (\lambda^2 - \mu^2 + 1)^2}{4\mu\lambda[(\lambda+\mu)^2 - 1]} \quad \text{when } \lambda^2 + \mu^2 \geq 1$$

If, on the other hand we take the $(-)$, then

$$\left(\frac{da}{ds}\right)^2 = \frac{4\lambda^2 - (\lambda^2 - \mu^2 + 1)^2}{4\mu\lambda\,[1 - (\lambda - \mu)^2]} \quad \text{when } \lambda^2 + \mu^2 < 1$$

Equations (10) and (11) then give a relation between

$\frac{da}{ds}$ and the positon of the object for the admissible ranges of λ and μ. The figures which follow give this relation in graphical form.

Figures 12 to 15

Figure 12 shows the variation of $\left(\frac{da}{ds}\right)^2$ as a function of one of the measured distance with the other distance as parameter, for the case

$$\lambda^2 + \mu^2 < 1$$

Figure 13 is similar to Figure 12 for the case

$$\lambda^2 + \mu^2 \geq 1$$

Figures 14 and 15 show a partitioning of the area under consideration by equal error contours in exactly the same way as was done in figure 6.

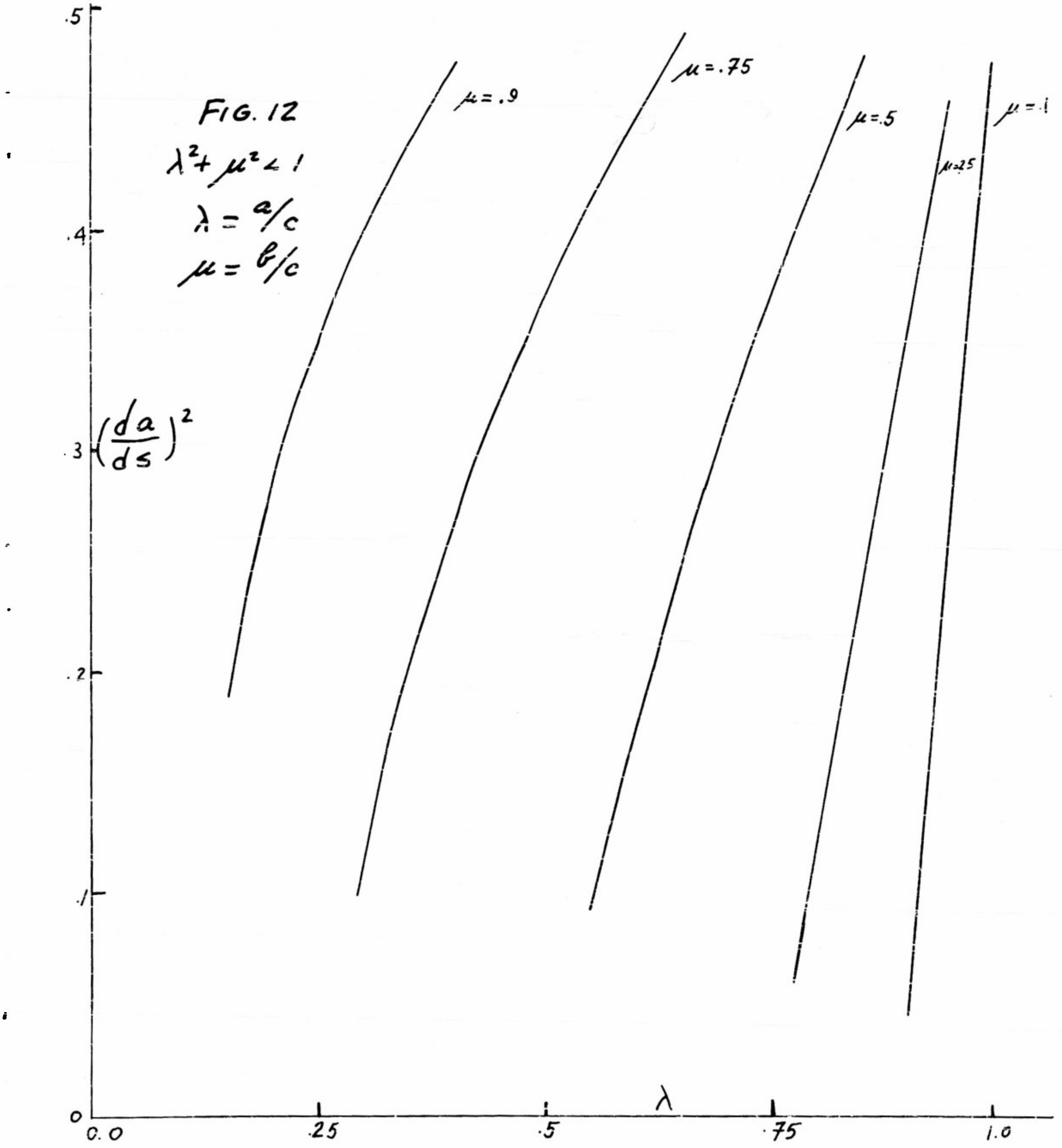


FIG. 12
$\lambda^2 + \mu^2 < 1$
$\lambda = a/c$
$\mu = b/c$
$\left(\frac{da}{ds}\right)^2$
.5
.4
.3
.2
.1
0
0.0
.25
.5
.75
1.0
λ
$\mu = .9$
$\mu = .75$
$\mu = .5$
$\mu = .25$
$\mu = .1$

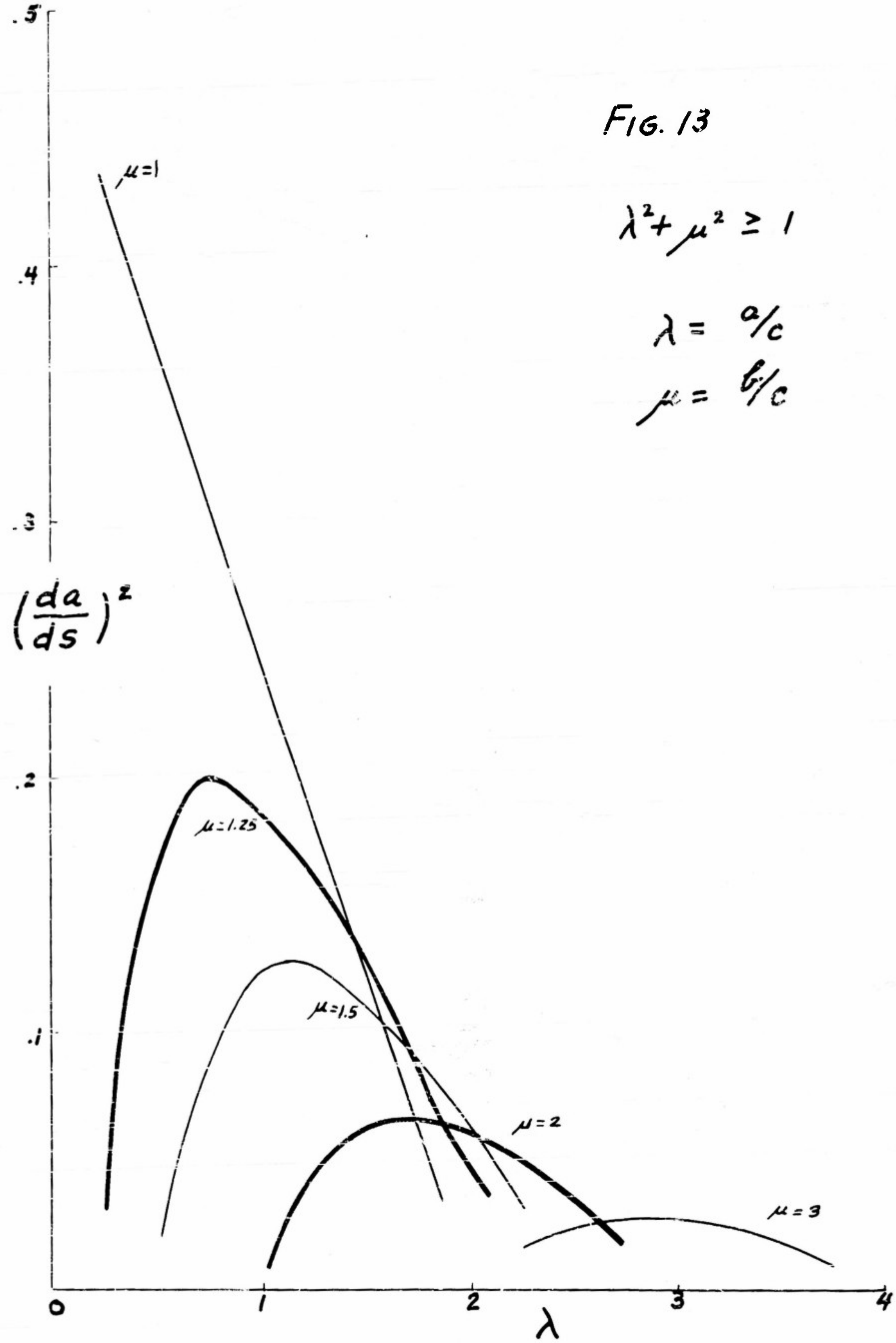


FIG. 13

$\lambda^2+\mu^2 \geq 1$

$\lambda = a/c$

$\mu = b/c$

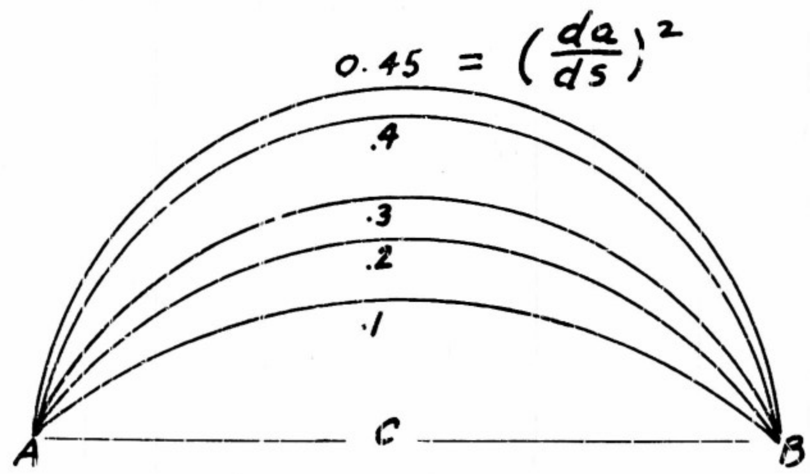


FIG. 14

$\lambda^2 + \mu^2 < 1$

$\lambda = a/c$

$\mu = b/c$

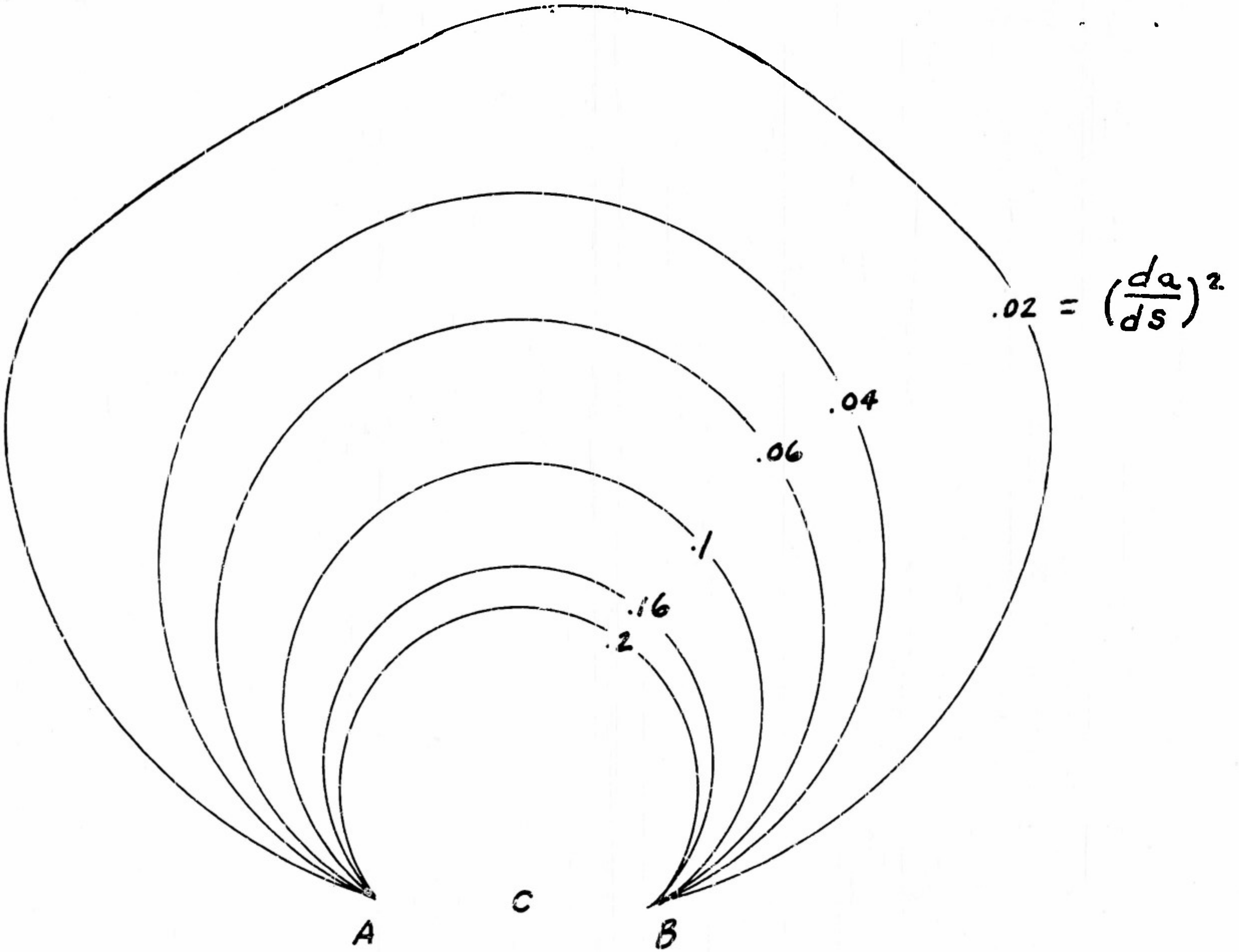


FIG. 15

$\lambda^2 + \mu^2 \geq 1$

$\lambda = a/c$

$\mu = b/c$

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